A Decomposition Method on Solving the Linear Arboricity Conjecture

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Abstract

A *linear forest* is a disjoint union of path graphs. The *linear arboricity of a graph G*, denoted by $\mathrm{la}(G)$, is the least number of linear forests into which the graph can be partitioned. Clearly, $\mathrm{la}(G) \geq \lceil \Delta(G)/2 \rceil$ for any graph of maximum degree $\Delta(G)$. For the upper bound, the long-standing *Linear Arboricity Conjecture* (LAC) due to Akiyama, Exoo, and Harary from 1981 asserts that $\mathrm{la}(G) \leq \lceil (\Delta(G)+1)/2 \rceil$. A graph is a *pseudoforest* if each of its component contains at most one cycle.

In this paper, we prove that *the union of any two pseudoforests of maximum degree up to 3 can be decomposed into three linear forests.* Combining it with a recent result of Wdowinski on the minimum number of pseudoforests into which that a graph can be decomposed, we prove that the LAC holds for the following simple graph classes: k-degenerate graphs with maximum degree $\Delta \geq 3k-1$, all graphs on nonnegative Euler characteristic surfaces provided the maximum degree $\Delta \neq 7$, and graphs on negative Euler characteristic ϵ surfaces provided the maximum degree $\Delta \geq 3\left\lceil \frac{5+\sqrt{49-24\epsilon}}{4} \right\rceil - 1$, as well as graphs with no K_t-minor satisfying some conditions on maximum degrees.

Keywords: f-coloring; pseudoforest; linear arboricity

1 Introduction

In this paper, unless otherwise specified, all graphs are simple, i.e., finite undirected graphs with no loops or multiple edges. Let G be an n-vertex graph with vertex set $V(G)$ and edge set $E(G)$. The *degree* of a vertex v in G, written $d_G(v)$ or $d(v)$, is the number of edges incident to v. Denote by $\Delta(G)$ the maximum degree of G. A *linear forest* is a union of vertex-disjoint paths. The *linear arboricity* of G, denoted by $\mathrm{la}(G)$, is the minimum number of linear forests needed to partition $E(G)$. Since it needs $\lceil \Delta(G)/2 \rceil$ linear forests to cover all edges incident to a maximum degree vertex, it follows that $\mathrm{la}(G) \geq \lceil \Delta(G)/2 \rceil$. In addition, it is easy to verify that

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$\mathrm{la}(G) \geq \left\lceil \frac{\Delta(G)+1}{2} \right\rceil$ provided G is regular – each path in G is of length at most $n-1$, which implies $\mathrm{la}(G) \geq \frac{|E(G)|}{n-1} = \frac{n \cdot \Delta(G)}{2(n-1)} > \frac{\Delta(G)}{2}$. In 1981, Akiyama, Exoo and Harary [2] conjectured that this lower bound for regular graph is an upper bound for any simple graph, which is commonly referred to as the *linear arboricity conjecture* (LAC).

Conjecture 1.1 (LAC)**.** *For every graph G,* $\mathrm{la}(G) \leq \left\lceil \frac{\Delta(G)+1}{2} \right\rceil$.

The LAC implies that $\lceil \Delta(G)/2 \rceil \leq \mathrm{la}(G) \leq \lceil (\Delta(G)+1)/2 \rceil$ for every graph G. An edge coloring of a graph is a partition of its edge set into matchings, which can be considered as a linear forest partition where each component is a single edge. Hence, the LAC can be viewed as an analog to Vizing's theorem [20] that $\Delta(G) \leq \chi'(G) \leq \Delta(G)+1$ for every simple graph G, where $\chi'(G)$ is the *chromatic index* of G. After nearly 30 years since the conjecture was posted, the famous LAC is still wide open although some progress has been made. The conjecture was verified for graphs with maximum degree $\Delta = 3, 4, 5, 6, 8, 10$. (See [1, 2, 7, 10].) It was confirmed for planar graphs by the combination of two papers [23, by Wu in 1999] and [25, by Wu and Wu in 2008]. Furthermore, Cygan, Hou, Kowalik, Lužar and Wu in 2011 [6] conjectured that the linear arboricity of a planar graph G is exactly $\lceil \Delta(G)/2 \rceil$ provided $\Delta(G) \geq 5$, and verified their conjecture under most cases, leaving open only the cases when $\Delta(G) = 6, 8$. Approximately and asymptotically, Alon in 1988 [3] proved that $\mathrm{la}(G) \leq \frac{\Delta(G)}{2} + O\left(\frac{\Delta(G) \log\log \Delta(G)}{\log \Delta(G)}\right)$. It was also showed in the same paper that the LAC holds for graphs with girth $\Omega(\Delta)$. Alon and Spencer in 1992 [4] further improved this bound. In 2019, Ferber, Fox and Jain [8] further narrowed it to $\Delta(G)/2 + O(\Delta(G)^{2/3-\alpha})$, where α is a positive constant. Recently, Lang and Postle [13] announced a better bound of $\Delta(G)/2 + 3\Delta(G)^{1/2} \log^4 \Delta(G)$. McDiarmid and Reed [16] confirmed the LAC for random regular graphs with fixed degrees. Glock, Kühn and Osthus [9] showed that, for a large range of p, a.a.s. the random graph $G \sim G_{n,p}$ can be decomposed into $\lceil \Delta(G)/2 \rceil$ linear forests.

A *pseudoforest* is a graph such that every component has at most one cycle. The *pseudoarboricity* of a graph G, denoted by $\mathrm{pa}(G)$, is the minimum number of spanning pseudoforests needed to partition $E(G)$. In fact, pseudoarboricity is defined as an analogy to the *arboricity* $\mathrm{a}(G)$, which is the minimum number of forests needed to partition $E(G)$. Note that each forest is a pseudoforest. Hence, $\mathrm{pa}(G) \leq \mathrm{a}(G)$. For a graph G and a function $f : V(G) \to \mathbb{N} \setminus \{0\}$, we call G a *degree-f pseudoforest* if it is a pseudoforest such that $d(v) \leq f(v)$ for any $v \in V(G)$. The *degree-f pseudoarboricity* $\mathrm{pa}_f(G)$ of G is the minimum number of degree-f spanning pseudoforests needed to partition $E(G)$. Wdowinski [22] obtained the exact value of $\mathrm{pa}_f(G)$ as follows.

Theorem 1.2. *For every multigraph G and function $f : V(G) \to \mathbb{N} \setminus \{0, 1\}$, we have* $\mathrm{pa}_f(G) = \max\{\Delta_f(G), \mathrm{pa}(G)\}$*, where* $\Delta_f(G) = \max\limits_{v \in V(G)} \left\lceil \frac{d(v)}{f(v)} \right\rceil$.

Wdowinski observed that every degree-4 pseudoforest can be decomposed into two linear forests, and so, $\mathrm{la}(G) \leq 2\,\mathrm{pa}_4(G)$. Below is our main result, whose proof is much more involved and deferred in Section 3.

Theorem 1.3. *If F_1, F_2 are two degree-3 pseudoforests, then the union graph $F_1 \cup F_2$ can be decomposed into three edge-disjoint linear forests.*

The following is an immediate corollary of Theorem 1.3.

Corollary 1.4. *For every graph G, we have* $\mathrm{la}(G) \leq \left\lceil \frac{3}{2}\,\mathrm{pa}_3(G) \right\rceil$.

Proof. Let $\mathrm{pa}_3(G) = k$. By definition, G can be partitioned into k degree-3 pseudoforests, say $F_1, \ldots, F_k$. If k is even, then we pair up these k degree-3 pseudoforests as $(F_1, F_2), (F_3, F_4), \ldots,$ (F_{k-1}, F_k). Otherwise, add a trivial spanning subgraph F_{k+1} of G with empty edge set, which is naturally a degree-3 pseudoforest. Pair up these $k+1$ pseudoforests as (F_1, F_2), (F_3, F_4), $\ldots$, (F_k, F_{k+1}). By Theorem 1.3, the union graph of each pair can be partitioned into three linear forests. Therefore, G can be partitioned into $\lceil 3k/2 \rceil$ linear forests, which implies $\mathrm{la}(G) \leq \lceil 3\,\mathrm{pa}_3(G)/2 \rceil$. $\square$

The following result shows that the LAC holds for graphs G with $\Delta(G) \geq 3\,\mathrm{pa}(G) - 1$.

Theorem 1.5. *For any graph G with $\Delta(G) \geq 3\,\mathrm{pa}(G) - 1$, we have* $\mathrm{la}(G) \leq \left\lceil \frac{\Delta(G)+1}{2} \right\rceil$.

For the sake of completing the proof of Theorem 1.5, the following two observations are needed, and the former was given by Wdowinski [22].

Observation 1.6. *Every loopless multigraph G has a linear forest F such that $\Delta(G - F) \leq \Delta(G) - 1$.*

Proof. Let $r = \Delta(G)$ or $\Delta(G) + 1$ be an even integer. Let G^* be an r-regular graph obtained from G by adding edges and vertices. Clearly, $G \subseteq G^*$. By Petersen's theorem [17], G^* has a 2-factor F^* – a spanning subgraph of G^* in which all vertices have degree two. Let F' be the graph obtained from F^* by removing the added edges and vertices. Then, each component of F' is either a path or cycle. Note that every vertex of degree $\Delta(G)$ in G must have degree at least 1 in F'. By arbitrarily removing an edge from each cycle component in F', we obtain a linear forest F also having the property that $d_F(v) \geq 1$ for every maximum degree vertex $v \in V(G)$. Hence, $\Delta(G - F) \leq \Delta(G) - 1$. $\square$

Observation 1.7. *For any $k \in \mathbb{N}$, we have*

$$\left\lceil \frac{3}{2} \left\lceil \frac{k-1}{3} \right\rceil \right\rceil + 1 = \left\lceil \frac{k+1}{2} \right\rceil \qquad \text{if } k \equiv 1 \bmod 6\text{; and}$$
$$\left\lceil \frac{3}{2} \left\lceil \frac{k}{3} \right\rceil \right\rceil \leq \left\lceil \frac{k+1}{2} \right\rceil \qquad \text{otherwise.}$$

These relationships can be easily checked, and with the results above, we are now ready to prove Theorem 1.5.

Proof of Theorem 1.5. Applying Theorem 1.2 with the constant function $f = 3$, we get the following equality.

$$\mathrm{pa}_3(G) = \max \left\{ \left\lceil \frac{\Delta(G)}{3} \right\rceil, \mathrm{pa}(G) \right\}. \tag{1}$$

Combining with Corollary 1.4, we have

$$\mathrm{la}(G) \leq \max \left\{ \left\lceil \frac{3}{2} \left\lceil \frac{\Delta(G)}{3} \right\rceil \right\rceil, \left\lceil \frac{3}{2} \mathrm{pa}(G) \right\rceil \right\}.$$

Since $\Delta(G) \geq 3\,\mathrm{pa}(G) - 1$, it follows that $\lceil \frac{3}{2} \mathrm{pa}(G) \rceil \leq \left\lceil \frac{\Delta(G)+1}{2} \right\rceil$. When $\Delta(G) \not\equiv 1 \bmod 6$, by Observation 1.7, we have $\left\lceil \frac{3}{2} \left\lceil \frac{\Delta(G)}{3} \right\rceil \right\rceil \leq \left\lceil \frac{\Delta(G)+1}{2} \right\rceil$, which implies that $\mathrm{la}(G) \leq \left\lceil \frac{\Delta(G)+1}{2} \right\rceil$.

Assume now that $\Delta(G) \equiv 1 \bmod 6$. Let $G' = G - E(F)$, where F is the linear forest guaranteed by Observation 1.6. Then, $\Delta(G') \leq \Delta(G) - 1$. Hence, by applying Theorem 1.2 and Corollary 1.4 to the graph G', we have

$$\begin{aligned}
\mathrm{la}(G) &\leq \mathrm{la}(G') + 1 \leq \lceil 3\,\mathrm{pa}_3(G')/2 \rceil + 1 \\
&= \max \left\{ \left\lceil \frac{3}{2} \left\lceil \frac{\Delta(G')}{3} \right\rceil \right\rceil + 1, \left\lceil \frac{3}{2} \mathrm{pa}(G') \right\rceil + 1 \right\} \\
&\leq \max \left\{ \left\lceil \frac{3}{2} \left\lceil \frac{\Delta(G)-1}{3} \right\rceil \right\rceil + 1, \left\lceil \frac{3}{2} \mathrm{pa}(G) \right\rceil + 1 \right\} \\
&= \max \left\{ \left\lceil \frac{\Delta(G)+1}{2} \right\rceil, \left\lceil \frac{3}{2} \mathrm{pa}(G) \right\rceil + 1 \right\}
\end{aligned}$$

Note that when $\Delta(G) \equiv 1 \bmod 6$, it is readily seen that $\Delta(G) \geq 3\,\mathrm{pa}(G) - 1$ is equivalent to $\Delta(G) \geq 3\,\mathrm{pa}(G)+1$, which in turn implies $\left\lceil \frac{\Delta(G)+1}{2} \right\rceil \geq \lceil \frac{3}{2} \mathrm{pa}(G) \rceil + 1$. Hence, $\mathrm{la}(G) \leq \left\lceil \frac{\Delta(G)+1}{2} \right\rceil$, which completes the proof of Theorem 1.5. □

2 Applications

For any positive integer k, a graph G is *k-degenerate* if it can be reduced to a trivial graph by successively removing vertices with degree at most k. The set of degenerate graphs contains many well-known families of graphs as special subclasses. For example, planar graphs are 5-degenerate.

The authors [5] previously proved that $\mathrm{la}(G) = \left\lceil \frac{\Delta(G)}{2} \right\rceil$ for k-degenerate graphs G with $\Delta(G) \geq 2k^2 - k$. For the slightly weaker maximum degree requirement, they showed that $\mathrm{la}(G) \leq \left\lceil \frac{\Delta(G)+1}{2} \right\rceil$. More precisely, they proved that the LAC holds for any k-degenerate graph G with $\Delta(G) \geq 2k^2 - 2k$. Wdowinski [22] recently proved that the LAC holds for any k-degenerate loopless multigraph G with $\Delta(G) \geq 4k-2$, which is a significant improvement of the second result above. Noticing that $3k-1 \leq 4k-2$ holds for every positive integer k, we improve Wdowinski's result as follows.

Theorem 2.1. *For any k-degenerate graph G with $\Delta(G) \geq 3k-1$, we have* $\mathrm{la}(G) \leq \left\lceil \frac{\Delta(G)+1}{2} \right\rceil$.

Proof. Note that an equivalent formulation of the k-degeneracy of G is that there exists an ordering of the vertices of G in which each vertex v is incident to at most k edges whose other endvertex precedes v in the ordering. Observe that we can distribute these edges, each incident to a vertex in the given ordering and having at most k edges connected to preceding vertices, into k distinct sets. Each set, by virtue of its construction, will not contain any cycles, and hence forms a forest. This collection of forests forms a partition of the graph G. (*We may simply say the following: At each vertex v, by distributing the edges incident with v and a vertex precedes v into k distinct sets, we obtain k 1-degenerate graphs. Notice that a graph is a forest if and only if it is 1-degenerate. Thus, a k-degenerate graph can be decomposed into k forests.*) Therefore, the arboricity $\mathrm{a}(G) \leq k$, which subsequently implies that the pseudoarboricity $\mathrm{pa}(G) \leq k$. Hence $\Delta(G) \geq 3\,\mathrm{pa}(G) - 1$. Thus, Theorem 2.1 follows immediately from Theorem 1.5. $\square$

Vizing [21] proved that every k-degenerate graph is class one, i.e. $\chi'(G) = \Delta(G)$, provided $\Delta(G) \geq 2k$. He further conjectured that there is a positive number ϵ such that if k is large enough, then every k-degenerate graph G with $\Delta(G) \geq (2-\epsilon)k$ is class one. Inspired by Vizing's work, we believe the following conjecture on linear forest partitions.

Conjecture 2.2. *For a k-degenerate graph G, if $\Delta(G) \geq 2k$, then* $\mathrm{la}(G) = \left\lceil \frac{\Delta(G)}{2} \right\rceil$.

Indeed, $\Delta(G) \geq 2k$ implies that $\mathrm{pa}(G) \leq k \leq \left\lceil \frac{\Delta(G)}{2} \right\rceil$. And by inputting the constant function $f = 2$ in Theorem 1.2, we reach the conclusion that $\mathrm{pa}_2(G) = \max\left\{ \left\lceil \frac{\Delta(G)}{2} \right\rceil, \mathrm{pa}(G) \right\} = \left\lceil \frac{\Delta(G)}{2} \right\rceil$. Therefore, we further conjecture that $\mathrm{la}(G) = \mathrm{pa}_2(G)$ when $\Delta(G) \geq 2k$, which may provide an approach on proving Conjecture 2.2.

As mentioned above, planar graphs are 5-degenerate. Directly applying Theorem 2.1 with $k = 5$, we see that the LAC holds for all planar graphs with maximum degree at least 14. We will show that the lower bound of maximum degree can be improved to 8. More generally, we consider graphs with no K_5-subdivision.

We say a graph G contains a graph H as a *minor* if H can be obtained by contracting edges in a subgraph of G. Analogously, one can define the topological minor which is actually used in the following context. A graph is called an H-subdivision if it can be formed by replacing some edges of a graph H with internally vertex-disjoint paths. We say that H is a *topological minor* of a graph G if G contains an H-subdivision as a subgraph. It is routine to check that every topological minor of a graph is also a minor. Kuratowski's theorem states that a finite graph is planar if and only if it does not contain a K_5-subdivision or a $K_{3,3}$-subdivision. Wagner proved planar graph has the same two graphs as forbidden minors. Mader [15] obtained the following result.

Theorem 2.3. *Let G be a simple graph. If G does not contain a K_5-subdivision, then $|E(G)| \leq 3|V(G)| - 6$.*

Hakimi [11] determined the exact value of the pseudoarboricity as follows.

Theorem 2.4. *For any graph G, we have*

$$\mathrm{pa}(G) = \max_{S \subseteq V(G), |S| \geq 1} \left\lceil \frac{e(S)}{|S|} \right\rceil,$$

where $e(S)$ is the number of edges with both endvertices in S.

As an application of Theorem 1.5, we show that the LAC holds for all graphs G with no K_5 subdivision except for $\Delta(G) = 7$.

Theorem 2.5. *Let G be a graph with no K_5-subdivision. If $\Delta(G) \neq 7$, then $\mathrm{la}(G) \leq \left\lceil \frac{\Delta(G)+1}{2} \right\rceil$.*

Proof. Let G be a graph without a K_5-subdivision and $\Delta(G) \neq 7$. As we mentioned at the beginning of the Introduction, the LAC holds for all graphs with maximum degree no more than 6. We assume that $\Delta(G) \geq 8$. By Mader's theorem, we have $|E(H)| \leq 3|V(H)| - 6$ for every subgraph H of G. By applying Hakimi's theorem, we have

$$\mathrm{pa}(G) = \max_{S \subseteq V(G), |S| \geq 1} \left\lceil \frac{e(S)}{|S|} \right\rceil \leq \max_{S \subseteq V(G), |S| \geq 1} \left\lceil \frac{3|S| - 6}{|S|} \right\rceil \leq 3.$$

Then, $3\,\mathrm{pa}(G) - 1 \leq 8 \leq \Delta(G)$. By Theorem 1.5, we have $\mathrm{la}(G) \leq \left\lceil \frac{\Delta(G)+1}{2} \right\rceil$. $\square$

Since planar graphs do not contain K_5-subdivision, the result above holds for all planar graphs with maximum degree $\Delta \neq 7$. We note that Theorem 1.5 can be applied to graphs G when $\mathrm{pa}(G)$ is small according to the formula given by Hakimi's theorem, which is particularly appealing when applied to graphs that are embedded in a surface with a fixed Euler characteristic.

The Euler characteristic ϵ of a surface (a compact, connected 2-manifold without boundary) Π is an integer-valued invariant of every homeomorphism type of surfaces. Let G be a connected graph on at least 3 vertices that has an embedding in the surface Π. Denote by $F(G)$ the set of faces of G. By the Euler characteristic formula, regardless of the way of embedding, we have

$$|V(G)| - |E(G)| + |F(G)| = \varepsilon.$$

Moreover, for any embedding of G, since every face has at least three edges on its boundary and each edge is counted at most twice when we sum up along all the face boundaries, it follows that $2|E(G)| \geq 3|F(G)|$. Combining this with the Euler characteristic formula, we have

$$|E(G)| \leq 3(|V(G)| - \epsilon). \tag{2}$$

Since inequality 2 applies to any subgraph of G, applying Hakimi's theorem we get $\mathrm{pa}(G) \leq 3$ provided $\epsilon \geq 0$. Following the exactly same proof, we have the following result.

Theorem 2.6. *For any graph G embedded on a surface with nonnegative Euler characteristic, we have* $\mathrm{la}(G) \leq \left\lceil \frac{\Delta(G)+1}{2} \right\rceil$ *provided* $\Delta(G) \neq 7$.

For $\epsilon < 0$, Wu [24] showed that $\mathrm{la}(G) = \left\lceil \frac{\Delta(G)}{2} \right\rceil$ when $\Delta(G) \geq \sqrt{46 - 54\varepsilon} + 19$. Here we confirm that the LAC is true for graphs with much smaller maximum degree.

Theorem 2.7. *For any graph G embedded on a surface with Euler characteristic $\epsilon < 0$, we have* $\mathrm{la}(G) \leq \left\lceil \frac{\Delta(G)+1}{2} \right\rceil$ *provided* $\Delta(G) \geq 3 \left\lceil \frac{5+\sqrt{49-24\epsilon}}{4} \right\rceil - 1$.

Proof. For any $S \subseteq V(G)$ with $|S| \geq 1$, we first show that $\frac{e(S)}{|S|} \leq \min\{3 - \frac{3\epsilon}{|S|}, \frac{|S|-1}{2}\}$. Inequality 2 implies that $\frac{e(S)}{|S|} \leq 3 - \frac{3\epsilon}{|S|}$. Noticing that there are at most $\binom{|S|}{2}$ edges in the graph induced by S, we have $\frac{e(S)}{|S|} \leq \frac{|S|-1}{2}$.

Let $f(x) = \min\{3 - \frac{3\epsilon}{x}, \frac{x-1}{2}\}$ be a function for any $x > 0$. Since $3 - \frac{3\epsilon}{x}$ is monotonic decreasing and $\frac{x-1}{2}$ is monotonic increasing, the maximum point x_0 of f satisfies $3 - \frac{3\epsilon}{x_0} = \frac{x_0 - 1}{2}$, so that $x_0 = \frac{7+\sqrt{49-24\epsilon}}{2}$. It follows that $f(x) \leq f(x_0) = \frac{5+\sqrt{49-24\epsilon}}{4}$. Therefore by Hakimi's theorem, we have

$$\mathrm{pa}(G) = \max_{S \subseteq V(G), |S| \geq 1} \left\lceil \frac{e(S)}{|S|} \right\rceil \leq \max_{S \subseteq V(G), |S| \geq 1} \lceil f(|S|) \rceil \leq \left\lceil \frac{5+\sqrt{49-24\epsilon}}{4} \right\rceil,$$

and so $3\,\mathrm{pa}(G) - 1 \leq \Delta(G)$. By Theorem 1.5, we have $\mathrm{la}(G) \leq \left\lceil \frac{\Delta(G)+1}{2} \right\rceil$. □

Even more generally, we can take the advantage of properties for graphs with forbidden minors, noticing that graphs on a fixed surface can be characterized by a finite set of forbidden minors by the Robertson-Seymour theorem. Let G be a graph with no K_t as a minor. When $3 \leq t \leq 9$, Mader [14], Jørgensen [12], and Song and Thomas [18] obtained that $|E(G)| < (t-2)|V(G)|$. Clearly, this inequality also holds for any subgraph induced by a vertex set $S \subseteq V(G)$, which implies that $\left\lceil \frac{e(S)}{|S|} \right\rceil \leq t - 2$

Again by applying Hakimi's theorem, we have $\mathrm{pa}(G) \le t-2$. Combining this with Theorem 1.5, we get the following result which generalizes Theorem 2.5 and completely confirms the LAC for graphs with no K_4 as a minor.

Theorem 2.8. *For $t \in \{3,4,\dots,9\}$, let G be a graph containing no K_t as a minor. If $\Delta(G) \ge 3t-7$, then $\mathrm{la}(G) \le \left\lceil \frac{\Delta(G)+1}{2} \right\rceil$.*

For large positive integer t, Thomason [19] proved that a graph G with no K_t as a minor satisfies

$$|E(G)| < (\alpha + o(1))t\sqrt{\ln t} \cdot |V(G)|,$$

where $\alpha = 0.319\dots$ is an explicit constant and $o(1)$ is a function of t tending to 0 as $t \to \infty$. It then follows that $\mathrm{pa}(G) \le (\alpha + o(1))t\sqrt{\ln t}$. Again by Hakimi's Theorem, we have that $\mathrm{pa}(G) \le (\alpha + o(1))t\sqrt{\ln t}$. Therefore, we may generally conclude as follows using Theorem 1.5.

Theorem 2.9. *There exists a constant $\alpha = 0.319\dots$ such that any graph G containing no K_t as a minor with $\Delta(G) \ge (3\alpha + o(1))t\sqrt{\ln t}$ satisfies $\mathrm{la}(G) \le \left\lceil \frac{\Delta(G)+1}{2} \right\rceil$.*

3 Proof of Theorem 1.3

We first introduce some notation that will be used in this section. Let G be a simple graph. The *complement* of G, denoted by $\overline{G}$, is the graph on the same vertex set $V(G)$ such that two distinct vertices of $\overline{G}$ are adjacent if and only if they are not adjacent in G. For any edge set $E^* \subseteq E(G) \cup E(\overline{G})$ regardless whether $E^* \subseteq E(G)$ or not, let $G \pm E^*$ be the graph obtained from G by adding or deleting all edges belong to E^*, and we write $G \pm e$ for $G \pm \{e\}$. Let $\langle E^* \rangle$ be the graph with vertex set $V(G)$ and edge set E^*.

Notice that $\langle E^* \rangle$ may have isolated vertices. A *matching* is a set of edges that share no ? common vertex. For vertex sets $U, W \subseteq V(G)$, let $E_G(U,W)$ denote the set of all edges of G joining a vertex of U with a vertex of W. The subgraph induced by $U \subseteq V(G)$, denoted by $G[U]$, is the subgraph whose vertex set is U and which contains precisely all edges of G with both endvertices in U.

The following technical result, which serves as the foundation of this paper, guarantees Theorem 1.3.

Theorem 3.1. *Let F be a degree-3 pseudoforest, and let $M^- \subseteq \overline{F}$ be a matching. Then, there exists a matching $M_F \subseteq F$ such that both $\langle M^- \cup M_F \rangle$ and $F - M_F$ are linear forests.*

let F_1 and F_2 be two degree-3 pseudoforests. We will show that $F_1 \cup F_2$ is a disjoint union of three linear forests.

Proof of Theorem 1.3. Assume, without loss of generality, that F_1 and F_2 are spanning subgraphs of a graph G. Note that $\emptyset$ is a trivial matching in $\overline{F_1}$. Applying Theorem 3.1 with

$F = F_1$ and $M^- = \emptyset$, there exists a matching $M_{F_1} \subseteq F_1$ such that both $\langle M^- \cup M_{F_1}\rangle$, i.e., $\langle M_{F_1}\rangle$, and $F_1 - M_{F_1}$ are linear forests. Note that $M_{F_1} \subseteq \overline{F_2}$ since $E(F_1) \cap E(F_2) = \emptyset$. Applying Theorem 3.1 again with $F = F_2$ and $M^- = M_{F_1}$, there exists a matching $M_{F_2} \subseteq F_2$ such that both $\langle M_{F_1} \cup M_{F_2}\rangle$ and $F_2 - M_{F_2}$ are linear forests. Thus, we have found the three edge-disjoint linear forests: $M_{F_1} \cup M_{F_2}$, $F_1 - M_{F_1}$ and $F_2 - M_{F_2}$, whose union of edge sets is $E(F_1 \cup F_2)$. $\square$

The remainder is dedicated to the proof of Theorem 3.1. From F, by adding new edges and/or vertices if necessary, there exists a degree-3 pseudoforest $F^* \supseteq F$ such that every component of F^* has a cycle and every vertex on the cycle has degree 3. For a given matching $M^- \subseteq \overline{F}$, if Theorem 3.1 holds for F^*, then it also holds for F because $F^* \supseteq F$ is a degree-3 pseudoforest and $M^- - E(F^*)$ is also a matching in $\overline{F^*}$. Therefore, for simplicity, we assume that each component of F has a cycle, and each vertex on the cycle has degree 3.

grammar?

Let $\mathcal{C}$ be the set of all cycles in F. Let $V_c = \bigcup_{C\in\mathcal{C}} V(C)$. For each $i \geq 0$, let $V_i = \{v \in V(F) : dist(v, V_c) = i\}$. Trivially, $V_0 = V_c$. Suppose that t is the maximum integer such that $V_t \neq \emptyset$, then obviously, $t \geq 1$. Let $F_i = F[V_0 \cup \cdots \cup V_i]$ for each $i \in [t]$, where $[t] := \{0, \ldots, t\}$. Clearly, $F_0 \subseteq F_1 \subseteq \cdots \subseteq F_t$, $E(F_i - F_{i-1}) = E_F(V_{i-1}, V_i)$ for each $i \neq 0$, and $C \subseteq F_0$ for any $C \in \mathcal{C}$. For any $v \in V_c$, we notice that v has exactly two neighbors on some cycle in $\mathcal{C}$ and one neighbor in V_1. Denote by v^* the unique neighbor of v in V_1.

Proposition 3.2. *There exists a matching $M_1 \subseteq F_1$ such that both $\langle M^- \cup M_1\rangle$ and $F_1 - M_1$ are linear forests.*

The proof of Proposition 3.2 is more convoluted and is placed in Subsection 3.2.

Proposition 3.3. *There exists a monotonic sequence of matchings $M_1 \subseteq M_2 \subseteq \cdots \subseteq M_t$ such that the following three properties hold for each $i \in \{1, 2, \ldots, t\}$.*

(i) $M_i \subseteq F_i$;

(ii) $\langle M^- \cup M_i\rangle$ *is a linear forest; and*

(iii) $F_i - M_i$ *is a linear forest.*

Since $M_t \subseteq F_t$ and $F_t = F$, the matching M_t in Proposition 3.3 is the desired matching M_F in Theorem 3.1. We first prove Proposition 3.3 by induction in Subsection 3.1.

3.1 Proof of Proposition 3.3

The existence of M_1 is guaranteed by Proposition 3.2. Suppose that we have constructed a monotonic sequence of matchings $M_1 \subseteq \cdots \subseteq M_i$ for some $i \in \{1, \ldots, t-1\}$ satisfying properties

How do we know $F_{i+1} - M_{i+1}$ is a Linear forest.

(i), (ii) and (iii) in Proposition 3.3. We construct M_{i+1} as follows based on the matching M_i.

Let $V_{i,3}$ be the set of degree 3 vertices in $F_{i+1} - M_i$. If $V_{i,3} = \emptyset$ then $d_{F_{i+1}-M_i}(v) \leq 2$ for any $v \in V(F_i)$. Let $M_{i+1} := M_i$. Note that $F_{i+1} = F_i \cup E_F(V_i, V_{i+1})$. It is easy to check that the matching M_{i+1} satisfies properties (i), (ii) and (iii) in Proposition 3.3.? Suppose now that $V_{i,3} \neq \emptyset$. We claim that $V_{i,3} \subseteq V_i$. Otherwise, let $v \in V_{i,3}$ such that $v \in V_j$ for some $j \in [i+1] \setminus \{i\}$. If $j = i+1$ then $d_{F_{i+1}}(v) = 1$, and so $d_{F_{i+1}-M_i}(v) \leq 1$; if $j \in [i-1]$, then $d_{F_{i+1}-M_i}(v) = d_{F_i - M_i}(v) \leq 2$. We reach a contradiction in either case. Let $V_{i,3} = \{v_1, \ldots, v_s\}$. Since $V_{i,3} \subseteq V_i$ and $i \geq 1$, each $v_h \in V_{i,3}$ has a unique neighbor in V_{i-1}, say u_h, and exactly two neighbors in V_{i+1}, say w_h, w'_h. Moreover, $w_1, w'_1, \ldots, w_s, w'_s$ are distinct. Since F is a degree-3 pseudoforest, $u_h v_h \notin M_i$. Let $F_{i,h} = F_i \cup \bigcup_{k=1}^{h} \{v_k w_k, v_k w'_k\}$ when $h \in \{1, 2, \ldots, s\}$ and $F_{i,0} = F_i$ for convention.

$v_1, v_2 \cdots v_h \in V_{i,3}$

and $d_{F_{i+1}-M_i}(v_h) = 3$ ..

Claim 3.4. *For each $h \in \{0, 1, \ldots, s\}$, there exists a matching $M_{i,h} \subseteq F_{i,h}$ such that $\langle M^- \cup M_{i,h} \rangle$ and $F_{i,h} - M_{i,h}$ are linear forests.*

Proof. Clearly, Claim 3.4 holds for $h = 0$ since we can let $M_{i,0} = M_i$. Suppose $h \geq 1$ and we have the desired matching $M_{i,h-1}$ such that $\langle M^- \cup M_{i,h-1} \rangle$ and $F_{i,h-1} - M_{i,h-1}$ are linear forests. By definition, $M_{i,h-1}$ does not contain any edges incident to vertex v_h. Consequently, all of v_h, w_h and w'_h have degree at most 1 in $\langle M^- \cup M_{i,h-1} \rangle$, which in turn implies that each of v_h, w_h and w'_h is either an isolated vertex or an endvertex of a path component of $\langle M^- \cup M_{i,h-1} \rangle$. Hence, not all of v_h, w_h and w'_h are in the same component of $\langle M^- \cup M_{i,h-1} \rangle$. We may assume without loss of generality that v_h and w_h are in different path components of $\langle M^- \cup M_{i,h-1} \rangle$, and we let $M_{i,h} = M_{i,h-1} \cup \{v_h w_h\}$. Clearly, $M_{i,h} \subseteq F_{i,h}$ and $M_{i,h}$ is a matching. Moreover, adding edge $v_h w_h$ to $\langle M^- \cup M_{i,h-1} \rangle$ does not create a cycle but combines two path components into a new path component. Therefore, both $\langle M^- \cup M_{i,h} \rangle$ and $F_{i,h} - M_{i,h}$ are linear forests. $\square$

Applying Claim 3.4 with $h = s$ and letting $M_{i+1} = M_{i,s}$, we thus find a matching $M_{i+1} \subseteq F_{i+1}$ such that $\langle M^- \cup M_{i+1} \rangle$ and $F_{i,s} - M_{i+1}$ are linear forests, and so $F_{i+1} - M_{i+1}$ is also a linear forest. This completes the proof of Proposition 3.3.

3.2 Proof of Proposition 3.2

This definition is independent from uv is an edge.

Let M be a matching of F_1 and uv be an edge of some cycle from $\mathcal{C}$. Denote by $E_{F_1}(u)$ and $E_{F_1}(v)$ the sets of three edges incident to u and v in F_1, respectively. Recall that u^* and v^* are the unique neighbors of u and v, respectively. Then, $\{uu^*, uv\} \subseteq E_{F_1}(u)$ and $\{vv^*, uv\} \subseteq E_{F_1}(v)$. If $M \cap (E_{F_1}(u) \cup E_{F_1}(v)) \subseteq \{uu^*, vv^*\}$, then we define the *$uv$-switch w.r.t. M* as the following operation: $M \longrightarrow M^*$, where $M^* := M + uv - uu^* - vv^*$. Clearly, M^* is still a matching of F_1.

Claim 3.5. *Let uv be an edge of a cycle in $\mathcal{C}$, and M be a matching of F_1 such that $M \cap (E_{F_1}(u) \cup E_{F_1}(v)) \subseteq \{uu^*, vv^*\}$. If u and v are in two different components D_u, D_v of $\langle M^- \cup M \rangle$, then after the uv-switch w.r.t. M, the union of D_u and D_v becomes a disjoint union of paths and all other components of $\langle M^- \cup M \rangle$ are unchanged.*

Proof. Note that each component of $\langle M^- \cup M \rangle$ is either a path or an even cycle, and so are D_u and D_v. Since $M \cap E_{F_1}(u) \subseteq \{uu^*\}$, regardless whether $uu^* \in D_u$ or not, $D_u - uu^*$ is either a path or a disjoint union of paths with u being an endvertex of a path. If $uu^* \notin D_u$, then $uu^* \notin M$, and so $E_{F_1}(u) \cap M = \emptyset$, which implies that D_u must be a path. Hence, $D_u - uu^*$ is either a single path or a disjoint union of paths with u being an endvertex of some path. The similar statement holds for $D_v - vv^*$. Hence, $D_u \cup D_v - uu^* - vv^* + uv$ is a disjoint union of paths. For any component D of $\langle M^- \cup M \rangle$, if $D \notin \{D_u, D_v\}$, by definition we have $E(D) \cap \{uu^*, vv^*, uv\} = \emptyset$ and $V(D) \cap \{u, v\} = \emptyset$, and so $D - uu^* - vv^* + uv$ are unchanged. $\square$

For a matching M of F_1, a cycle $C \in \mathcal{C}$ is said to be M-*good* if $E(C) \cap M \neq \emptyset$ and each component of $\langle M^- \cup M \rangle$ sharing a common vertex with C is a path. Otherwise, C is said to be M-*bad*, i.e., $E(C) \cap M = \emptyset$ or there is a cycle component D of $\langle M^- \cup M \rangle$ such that $V(D) \cap V(C) \neq \emptyset$. A matching M of F_1 is called a *feasible matching* if it satisfies

(i) $\Delta(F_1 - M) \leq 2$; and

(ii) $E(C) \cap M = \emptyset$ for every M-bad cycle C.

We notice that under the condition (i), the condition (ii) is equivalent to that $M \cap E_{F_1}(u) = \{uu^*\}$ for every $u \in V(C)$ whenever C is M-bad. And so we can conduct a uv-switch w.r.t. M for every $uv \in E(C)$ whenever C is M-bad.

We start with the matching $M := \{vv^* : v \in V_c\}$. Then, $\Delta(F_1 - M) = 2$ and $E(C) \cap M = \emptyset$ for every cycle $C \in \mathcal{C}$. By definition, every cycle of $\mathcal{C}$ is M-bad. Hence, M is a feasible matching, and so the set of feasible matchings is not empty. For any two feasible matchings M_a and M_b in F_1, we write $M_a \prec M_b$ if every M_a-good cycle is M_b-good but not every M_b-good cycle is M_a-good. Clearly, all feasible matchings under the order $\prec$ form a poset. Let M_1 be a maximal element in this poset.

Proposition 3.6. *All cycles in F_1 are M_1-good.*

Suppose that Proposition 3.6 holds. Then, $E(C) \cap M_1 \neq \emptyset$ for any cycle C in F_1, and so $F_1 - M_1$ is acyclic. Since M_1 is a feasible matching, $\Delta(F_1 - M_1) \leq 2$. Hence, $F_1 - M_1$ is a linear forest. Let D be a nontrivial component of $\langle M^- \cup M_1 \rangle$. Notice that $\langle M^- \cup M_1 \rangle$ is a disjoint union of paths and cycles. We claim that D is a path. Recall that V_c is the set of vertices of all cycles in F. If $V(D) \cap V_c \neq \emptyset$, then D is a path because all cycles in F_1 are M_1-good. If $V(D) \cap V_c = \emptyset$, then $E(D) \cap M_1 = \emptyset$ because every edge of $M_1 \subseteq E(F_1)$ is incident to some vertex in V_c. Hence, D is a single edge of M^-, and so D is also a path. Accordingly, $\langle M^- \cup M_1 \rangle$ is a disjoint union of paths, which also shows that $\langle M^- \cup M_1 \rangle$ is a linear forest. So far, we have completed the proof of Proposition 3.2 under Proposition 3.6. The remainder is devoted to the proof of Proposition 3.6. We assume that there is an M_1-bad cycle and will reach a contradiction to complete the proof.

? We do not need D being a directed path or cycle. We only need an orientation.

For a nontrivial component D of $\langle M^- \cup M_1 \rangle$, noticing that D is either a path or an even cycle, we denote an orientation of D by $\prec_D$ as if D is a directed path or cycle. For a vertex $v \in V(D)$, let v^-, v^+ denote the predecessor and successor of v along the orientation $\prec_D$ if they exist. For any $u, v \in V(D)$, denote by $D[u, v]$ the subpath of D from u to v along the orientation $\prec_D$, and denote by $D^-[u, v]$ the subpath of D from u to v along the reversed orientation of $\prec_D$. A *chord* is an edge that is not part of a cycle (resp. path) but connects two vertices of a cycle (resp. path). Two chords xy and zw of D are *crossing* if exactly one of z and w, say z, is such that $x \prec_D z \prec_D y$. Let $C \subseteq F_1$ be an M_1-bad cycle. Suppose that there is an edge $uv \in E(C)$ with endvertices $u, v \in V(D)$. Since M_1 is a feasible matching, we have $M_1 \cap (E_{F_1}(u) \cup E_{F_1}(v)) = \{uu^*, vv^*\}$, and so $uu^*, vv^* \in E(D)$, $uv \notin E(D)$. We call such uv a *bad-cycle-chord* (b-chord) of D. When D is a cycle, we say that uv is a *consistent b-chord* if either $u^+ = u^*$ and $v^- = v^*$, or $u^- = u^*$ and $v^+ = v^*$. When D is a path, we assume $u \prec_D v$, and say that uv is a *consistent b-chord* if $u^- = u^*$ and $v^+ = v^*$. Otherwise, we call uv an *inconsistent b-chord*.

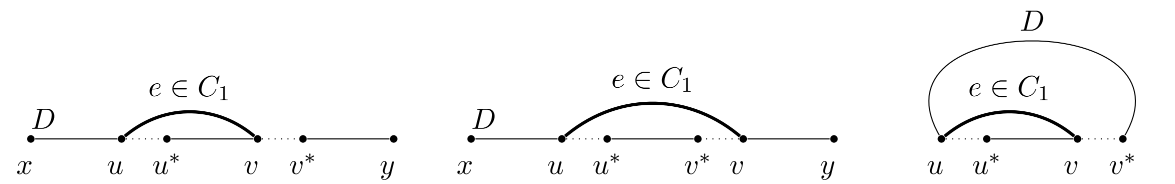
Claim 3.7. *Let D be a nontrivial component of $\langle M^- \cup M_1 \rangle$. Then, the following statements hold.*

(i) For any M_1-bad cycle C_1, if $V(C_1) \subseteq V(D)$, then every edge of $E(C_1)$ is a consistent b-chord of D.

(ii) For any two M_1-bad cycles C_1 and C_2, if $V(C_1) \cup V(C_2) \subseteq V(D)$, then no two edges of $E(C_1) \cup E(C_2)$ form a pair of crossing b-chords of D.

Proof. For (i): Suppose on the contrary that there is an edge $e = uv \in E(C_1)$ that is an inconsistent b-chord of D. When D is a path, we assume that x, y are the two endvertices of D and $x \prec_D u \prec_D v \prec_D y$. By symmetry, it suffices to consider the following three cases: (1) $u^+ = u^*$ and $v^+ = v^*$ when D is a path (Figure 1a); (2) $u^+ = u^*$ and $v^- = v^*$ when D is a path (Figure 1b); or (3) $u^+ = u^*$ and $v^+ = v^*$ when D is a cycle (Figure 1c). In any of these cases, after conducting a uv-switch w.r.t. M_1, the only changed component D containing $V(C_1)$ becomes the union of paths D^* in $\langle M^- \cup M_1^* \rangle$, where $M_1^* := M_1 + uv - uu^* - vv^*$ and $D^* := D + uv - uu^* - vv^*$.

More precisely, D^* consists of two paths in case (1): $D[x,u]uvD^-[v,u^*]$ and $D[v^*,y]$; D^* consists of two paths in case (2): $D[x,u]uvD[v,y]$ and $D[u^*,v^*]$; $D^* = D[u^*,v]vuD^-[u,v^*]$ is a path in case (3). Combining this with $E(C^*) \cap M_1^* = E(C^*) \cap M_1$ for any $C^* \in \mathcal{C}\backslash\{C_1\}$, we have the following two results: C^* is M_1^*-good if C^* is M_1-good; C^* is M_1-bad if C^* is M_1^*-bad. Since M_1 is feasible matching, $\Delta(F_1 - M_1) \leq 2$ and $E(C^*) \cap M_1 = \emptyset$ if C^* is M_1-bad. So $\Delta(F_1 - M_1^*) = \Delta(F_1 - M_1 - uv + uu^* + vv^*) \leq 2$ and $E(C^*) \cap M_1^* = \emptyset$ if C^* is M_1^*-bad. Hence, M_1^* is a feasible matching with $M_1 \prec M_1^*$. Since $V(D^*) = V(D)$, we have $V(C_1) \subseteq V(D^*)$, and so each component of $\langle M^- \cup M_1^* \rangle$ sharing a common vertex with C_1 is a path. Combining this with $uv \in E(C_1) \cap M_1^*$, we conclude that C_1 is M_1^*-good, which gives a contradiction to the maximality of M_1.



(a) D is a path in case (1) (b) D is a path in case (2) (c) D is a cycle in case (3)

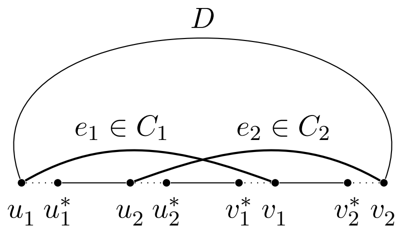
Figure 1: Inconsistent b-chords of D

For (ii): Suppose on the contrary that there are two crossing b-chords $e_1 = u_1v_1 \in E(C_1)$ and $e_2 = u_2v_2 \in E(C_2)$. We further assume that $C_1 = C_2$ if e_1, e_2 are on the same cycle. After conducting the u_1v_1-switch and the u_2v_2-switch, let

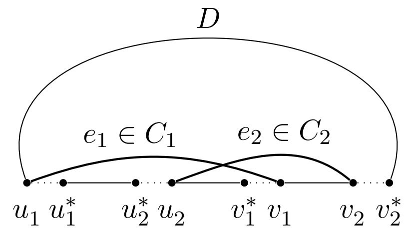
$$M_1^* := M_1 + u_1v_1 + u_2v_2 - u_1u_1^* - v_1v_1^* - u_2u_2^* - v_2v_2^*,$$
$$D^* := D + u_1v_1 + u_2v_2 - u_1u_1^* - v_1v_1^* - u_2u_2^* - v_2v_2^*.$$

Suppose that D is a cycle, and without loss of generality, that $u_1 \prec_D u_2 \prec_D v_1 \prec_D v_2$. By (i), both u_1v_1 and u_2v_2 are consistent b-chords of D. By symmetry, we assume that $u_1^+ = u_1^*$ and $v_1^- = v_1^*$, and consider the following two possible cases: (1) $u_2^+ = u_2^*$ and $v_2^- = v_2^*$ (Figure 2a); or (2) $u_2^- = u_2^*$ and $v_2^+ = v_2^*$ (Figure 2b). In case (1), D^* is a union of two paths: $D[u_2^*, v_1^*]$ and $D[u_1^*, u_2]u_2v_2D[v_2, u_1]u_1v_1D[v_1, v_2^*]$. In case (2), D^* is also a union of two paths: $D[u_1^*, u_2^*]$ and $D^-[v_1^*, u_2]u_2v_2D^-[v_2, v_1]v_1u_1D^-[u_1, v_2^*]$. Similarly to the discussion in Claim 3.7-(i), we have that $M_1 \prec M_1^*$ and M_1-bad cycle C_1 becomes M_1^*-good, giving a contradiction to the maximality of M_1.

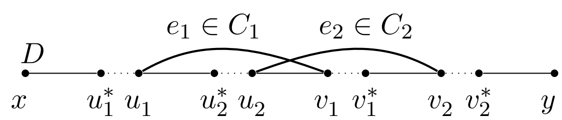
We now assume that D is a path. Let x, y be two endvertices of D. We assume without loss of generality that $x \prec_D u_1 \prec_D u_2 \prec_D v_1 \prec_D v_2 \prec_D y$. Since both u_1v_1 and u_2v_2 are consistent w.r.t. D, we have $u_1^- = u_1^*$, $v_1^+ = v_1^*$, $u_2^- = u_2^*$ and $v_2^+ = v_2^*$ (Figure 2c). We see that D^* is a union of three disjoint paths: $D[x, u_1^*]$, $D[v_2^*, y]$, and $D^-[u_2^*, u_1]u_1v_1D^-[v_1, u_2]u_2v_2D^-[v_2, v_1^*]$.



(a) D is a cycle in case (1)



(b) D is a cycle in case (2)



(c) D is a path

Figure 2: Crossing b-chords of D

Similarly to the discussion in Claim 3.7-(i), we have that $M_1 \prec M_1^*$ and the M_1-bad cycles C_1, C_2 become M_1^*-good, giving a contradiction to the maximality of M_1. $\square$

A cycle component D of $\langle M^- \cup M_1 \rangle$ is called a *black hole* if $V(C) \subseteq V(D)$ for any cycle $C \in \mathcal{C}$ satisfying $V(C) \cap V(D) \neq \emptyset$ with at most one exception $|V(C) \cap V(D)| = 1$. Since we assumed that D is a cycle in the above definition, every cycle sharing a vertex with a black hole is M_1-bad.

Claim 3.8. *There is no black hole in $\langle M^- \cup M_1 \rangle$.*

Proof. Suppose on the contrary that there is a black hole D in $\langle M^- \cup M_1 \rangle$. Let C_0 denote the only possible cycle that shares exactly one vertex with D, and let $\mathcal{B}^*$ denote the set of cycles $C^* \in \mathcal{C}$ such that $V(C^*) \subseteq V(D)$. Since D is a cycle component of $\langle M^- \cup M_1 \rangle$, it follows that D contains at least two edges of M_1, i.e., $|E(D) \cap M_1| \geq 2$. Since $M_1 \subseteq F_1$ and $F_1 = F[V_c \cup V_1]$, we have $|V(D) \cap V_c| \geq 2$ which in turn shows $\mathcal{B}^* \neq \emptyset$.

Let $E(\mathcal{B}^*) := \cup_{C^* \in \mathcal{B}^*} E(C^*)$. We define an auxiliary graph G with $V(G) = V(D)$ and $E(G) = E(D) \cup E(\mathcal{B}^*)$. Recall that every cycle in $\mathcal{C}$ sharing a vertex with a black hole is M_1-bad. Since M_1 is feasible matching, every edge in $E(\mathcal{B}^*)$ is a b-chord of D. By Claim 3.7-(ii), no two edges of $E(\mathcal{B}^*)$ are crossing w.r.t. D. Hence, G can be drawn on the plane as an outerplanar graph, where D is the boundary and all edges in $E(\mathcal{B}^*)$ are chords of G (Figure 3).

If C_0 exists, then let v_0 be the unique vertex in $V(C_0) \cap V(D)$, otherwise, let v_0 be an arbitrary vertex not incident to any $e \in E(\mathcal{B}^*)$, which exists by outerplanarity. Each edge $e \in E(\mathcal{B}^*)$ divides the interior region of D into two parts: R_e^1 whose closure contains v_0, and R_e^0

Since every 2-connected outerplanar graphs have a vertex of degree 2.

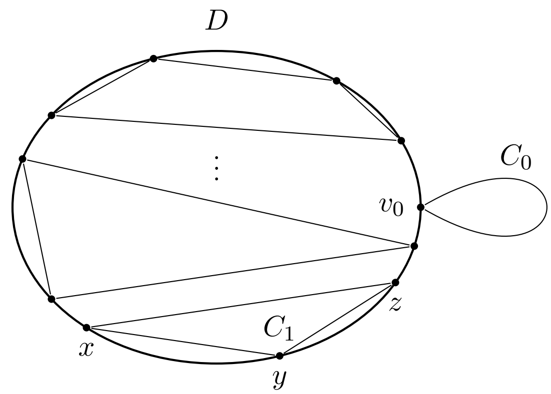


Figure 3: An illustration of the outerplanar graph G

whose closure does not contain v_0. Of all edges on $E(\mathcal{B}^*)$, let e_1 be one such that $R^1_{e_1}$ contains most number of cycles in $\mathcal{B}^*$, and let $C_1 \in \mathcal{B}^*$ be the cycle containing e_1. We claim that for any edge $e \in E(C_1)$, R^0_e contains no cycle in $\mathcal{B}^*$ other than C_1 itself. Otherwise, we suppose that there exists some edge $e_2 \in E(C_1)$ such that $R^0_{e_2}$ contains another cycle $C_2 \in \mathcal{B}^*$ other than C_1 itself. Then, for any edge $e_3 \in C_2$, $R^1_{e_3}$ contains more cycles than $R^1_{e_1}$, giving a contradiction.

Since C_1 has at least three edges, C_1 contains two consecutive edges xy and yz such that both R^0_{xy} and R^0_{yz} contain no cycles of $\mathcal{B}^*$. Assume without loss generality that $x \prec_D y \prec_D z$. Since M_1 is feasible and C_1 is M_1-bad, we have $M_1 \cap E_{F_1}(u) = \{uu^*\}$ for every $u \in V(C_1)$, and so $xx^*, yy^*, zz^* \in E(D)$. By Claim 3.7-(i), both xy and yz are consistent. Then, either $x^- = x^*$ and $y^+ = y^*$, or $x^+ = x^*$ and $y^- = y^*$. Suppose first that $x^- = x^*$ and $y^+ = y^*$. Recall that $xx^*, yy^* \in M_1$. Since D is a component of $\langle M^- \cup M_1 \rangle$, we have that $xx^+, yy^- \in M^-$. Since xy is a chord of D, it follows that $x^+ \neq y$, and so $xx^+ \neq yy^-$. Hence, $D[x^+, y^-]$ contains at least one edge of M_1, which in turn shows that it intersects with some cycle in $\mathcal{C}$. Therefore, there is a cycle of $\mathcal{B}^*$ in R^0_{xy}, giving a contradiction. We now suppose that $x^+ = x^*$ and $y^- = y^*$. Then, $z^+ = z^*$. In the same fashion above, we can show that R^0_{yz} contains a cycle of $\mathcal{B}^*$, giving a contradiction. $\square$

Claim 3.9. *If D is a path component in $\langle M^- \cup M_1 \rangle$, then $V(C) \setminus V(D) \neq \emptyset$ for any M_1-bad cycle C.*

Proof. Suppose on the contrary that there exists an M_1-bad cycle C such that $V(C) \subseteq V(D)$. By Claim 3.7-(ii), no two edges of C are crossing b-chords w.r.t. D. Note that C has at least three edges. Thus there are two consecutive edges $uv, vw \in E(C)$ such that $u \prec_D v \prec_D w$. By Claim 3.7-(i), both uv and vw are consistent w.r.t. D. However, when uv is consistent, we have $u^- = u^*$ and $v^+ = v^*$, which implies that vw is not consistent, giving a contradiction. $\square$

A cycle $C \in \mathcal{C}$ is a *single-point-unchangeable* (SPU) if the following is true: for any component

D in $\langle M^- \cup M_1 \rangle$, if $V(D) \cap V(C) \neq \emptyset$ then D is a cycle and $|V(D) \cap V(C)| = 1$, i.e., components of $\langle M^- \cup M_1 \rangle$ sharing a common vertex with C are cycles, and each of them shares exactly one vertex of C.

Claim 3.10. *If an M_1-bad cycle C intersects with at least two components of $\langle M^- \cup M_1 \rangle$, then C is an SPU.*

Proof. Suppose on the contrary that C is not an SPU. Let $C = u_0 u_1 \cdots u_s u_0$, and for convention, let $u_{s+1} = u_0$. Since C is an M_1-bad cycle and M_1 is a feasible matching, $M_1 \cap E_{F_1}(u_i) = \{u_i u_i^*\}$ for each $i \in \{0, \ldots, s\}$ and the $u_i u_{i+1}$-switch can be conducted. We will identify a set of independent edges $u_i u_{i+1}$ for $i \in \{1, \ldots, s\}$, and then do $u_i u_{i+1}$-switches for all these edges to get a feasible matching M_1^* such that $M_1 \prec M_1^*$ and C is M_1^*-good cycle, giving a contradiction to the maximality of M_1. Let $\mathcal{D}$ be the set of all components of $\langle M^- \cup M_1 \rangle$ sharing at least one vertex with C. By assumption, we have $|\mathcal{D}| \geq 2$.

We first consider the case that all components in $\mathcal{D}$ are paths. Since $|\mathcal{D}| \geq 2$, we may assume that u_0 and u_1 are contained in two different path components $D_0, D_1 \in \mathcal{D}$. We do $u_0 u_1$-switch and let $M_1^* = M_1 + u_0 u_1 - u_0 u_0^* - u_1 u_1^*$. By Claim 3.5, every component of $\langle M^- \cup M_1^* \rangle$ sharing a vertex with C is still a path, while after the switch, C is M_1^*-good since $E(C) \cap M_1^* = \{u_0 u_1\}$. For any M_1-good cycle $C^* \in \mathcal{C} \backslash \{C\}$, C^* is M_1^*-good since $E(C^*) \cap M_1^* = E(C^*) \cap M_1$. Moreover, since the $u_0 u_1$-switch changes no other M_1-bad cycles, it is easy to check that M_1^* is a feasible matching, and therefore, $M_1 \prec M_1^*$. This contradicts to the maximality of M_1.

Suppose now that there exists one cycle component in $\mathcal{D}$. Denote by $\mathcal{D}_C$ the set of all cycle components in $\mathcal{D}$. Since C is not an SPU, $\mathcal{D}$ either contains a path, or contains a cycle that shares at least two vertices with C. So, we assume that $D_0 \in \mathcal{D}$ is a path if there is one, otherwise is a cycle satisfying $|V(D_0) \cap V(C)| \geq 2$. In both cases, we assume that $u_0 \in V(D_0)$ and $u_s \notin V(D_0)$ because C intersects at least two components in $\mathcal{D}$.

We claim that there exists some index $i \in \{1, \ldots, s\}$ such that u_i belongs to a cycle component and u_{i+1} belongs to a different component which may be a path or a cycle. If D_0 is a path, then such i exists and $1 \leq i \leq s$ because of $\mathcal{D}_C \neq \emptyset$ and $u_s \notin V(D_0)$; if D_0 is a cycle, then such i exists and $1 \leq i \leq s-1$ because of $|V(D_0) \cap V(C)| \geq 2$ and $u_s \notin V(D_0)$. Let i_1 be the smallest index with above property.

Suppose that we have picked vertices $u_{i_1}, \ldots, u_{i_t}$ for $t \geq 1$ with above property. For each $j \in \{1, \ldots, t\}$, we denote the corresponding cycle component that u_{i_j} belongs to by D_j and the cycle or path component that u_{i_j+1} belongs to by D_i^*. If $\mathcal{D}_C \setminus \{D_1, D_1^*, \ldots, D_t, D_t^*\} = \emptyset$, then we stop. Otherwise, let i_{t+1} be the smallest index in $\{i_t + 2, \ldots, s\}$ such that $u_{i_{t+1}}$ is on a cycle component in $\mathcal{D}_C \setminus \{D_1, \ldots, D_t\}$ and $u_{i_{t+1}+1}$ belongs to a different component. We assume without loss of generality that $u_{i_{t+1}} \in V(D_{t+1})$ and $u_{i_{t+1}+1} \in V(D^*_{t+1})$. Assume this process terminates at u_{i_ℓ}. Then, $\{D_1, D^*_1, \ldots, D_\ell, D^*_\ell\} \supseteq \mathcal{D}_C$ and the cycles $D_1, \ldots, D_\ell$ are mutually distinct. Recall that $1 \le i_\ell \le s$, and so edges set $\{u_{i_j}u_{i_j+1} : 1 \le j \le \ell\}$ are independent.

We do $u_{i_1}u_{i_1+1}$-switch, ..., $u_{i_\ell}u_{i_\ell+1}$-switch w.r.t. M_1 and let $M_1^* := M_1 - \bigcup_{j=1}^{\ell}\{u_{i_j}u^*_{i_j}, u_{i_j+1}u^*_{i_j+1}\} + \bigcup_{j=1}^{\ell}\{u_{i_j}u_{i_j+1}\}$. Since edges set $\{u_{i_j}u_{i_j+1} : 1 \le j \le \ell\}$ are independent and $M_1 \cap (E_{F_1}(u_{i_j}) \cup E_{F_1}(u_{i_j+1}) = \{u_{i_j}u^*_{i_j}, u_{i_j+1}u^*_{i_j+1}\}$, we have that M_1^* is a matching of F_1. By applying Claim 3.5 repeatedly, we see that the subgraph of $\langle M^- \cup M_1^* \rangle$ induced by $\bigcup_{j=1}^{\ell}\left(V(D_{i_j}) \cup V(D^*_{i_j})\right)$ is a disjoint union of paths. So, all components in $\langle M^- \cup M_1^* \rangle$ intersecting with C are paths. Combining this with $\bigcup_{j=1}^{\ell}\{u_{i_j}u_{i_j+1}\} \subseteq E(C) \cap M_1^*$, we have that C is M_1^*-good. Note that $E(C^*) \cap M_1^* = E(C^*) \cap M_1$ for any cycle $C^* \in \mathcal{C}\backslash\{C\}$. Similarly to the discussion in the first case, we can verify that M_1^* is feasible matching with $M_1 \prec M_1^*$, giving a contradiction to the maximality of M_1. $\square$

out of space.

Now we are ready to complete the proof of Proposition 3.6 that all cycles in F_1 are M_1-good.

Proof of Proposition 3.6: Suppose on the contrary that there exists some M_1-bad cycles in F_1. Let $\mathcal{B}$ be the set of all M_1-bad cycles in F_1. Let

$$\mathcal{B}_1 = \{C \in \mathcal{B} : C \text{ is an SPU}\}, \text{ and}$$
$$\mathcal{B}_2 = \{C \in \mathcal{B} : V(C) \subseteq V(D) \text{ for some component } D \text{ of } \langle M^- \cup M_1 \rangle\}.$$

By Claim 3.10, we have $\mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2$ and $\mathcal{B}_1 \cap \mathcal{B}_2 = \emptyset$. By assumption, $\mathcal{B} \neq \emptyset$. We now consider the following cases on whether $\mathcal{B}_1 = \emptyset$ or not.

Case 1: $\mathcal{B}_1 = \emptyset$. In this case, $\mathcal{B}_2 = \mathcal{B} \neq \emptyset$. Let $C \in \mathcal{B}_2$ and D be the component of $\langle M^- \cup M_1 \rangle$ such that $V(C) \subseteq V(D)$. By Claim 3.9, we see that D is a cycle. By definition, any cycle $C^* \in \mathcal{C}$ having nonempty intersection with D is M_1-bad, i.e., $C^* \in \mathcal{B}$. And so $C^* \in \mathcal{B}_2$ because of $\mathcal{B}_1 = \emptyset$, which in turn implies that $V(C^*) \subseteq V(D)$. Hence, D is a black hole, giving a contradiction to Claim 3.8.

Case 2: $\mathcal{B}_1 \neq \emptyset$. Define an auxiliary multi-hypergraph H with $V(H) = \mathcal{B}_1$; and for each component D of $\langle M^- \cup M_1 \rangle$ intersecting with at least two SPUs, we define an edge E_D of H by $E_D = \{C \in \mathcal{B}_1 : V(C) \cap V(D) \neq \emptyset\}$. A **Berge cycle** of length ℓ in a hypergraph is a set of ℓ distinct vertices $\{v_1, \ldots, v_\ell\}$ and ℓ distinct edges $\{e_1, \ldots, e_\ell\}$ such that $\{v_i, v_{i+1}\} \subseteq e_i$.

We claim that H contains a Berge cycle. Suppose on the contrary that H does not contain a Berge cycle. Then, there exists some vertex $C \in V(H)$ such that $d_H(C) \le 1$, i.e., there exists at most one component of $\langle M^- \cup M_1 \rangle$ that intersects C and another SPU. Denote such a component by D_0 if exists. Note that $|V(C) \cap V(D_0)| = 1$. Since $M_1 \cap E_{F_1}(v) = \{vv^*\}$ for any $v \in V(C)$, there exists another component, say D, of $\langle M^- \cup M_1 \rangle$ intersecting with C. Since C

is an SPU, it follows that $|V(C) \cap V(D)| = 1$ and D is a cycle. Hence, for any $C^* \in \mathcal{B} \setminus \{C\}$, if $V(C^*) \cap V(D) \neq \emptyset$ then C^* is M_1-bad. If $C^* \in \mathcal{B}_1$, then E_D is an edge of H by definition, which in turn shows $d_H(C) \geq 2$, giving a contradiction. Thus, $C^* \in \mathcal{B}_2$, which in turn gives $V(C^*) \subseteq V(D)$. Therefore, D is a black hole, giving a contradiction to Claim 3.8.

Let $B = C_1 C_2 \cdots C_m C_1$ be a Berge cycle of length m in H, and let $E_{D_1}, \ldots, E_{D_m}$ be the edges in $E(H)$ such that $\{C_i, C_{i+1}\} \subseteq E_{D_i}$ for $i \in \{1, \ldots, m\}$ where $C_{m+1} = C_1$. Since both C_i and C_{i+1} are SPUs, we have that D_i is a cycle and $|V(C_i) \cap V(D_i)| = |V(C_{i+1}) \cap V(D_i)| = 1$. Let $v_i \in V(C_i) \cap V(D_i)$ and $w_i \in V(C_{i+1}) \cap V(D_i)$. Then, $v_1, w_1, \ldots, v_m, w_m$ are distinct vertices.

Let $C_1 = v_1 x_1 \ldots x_t v_1$. For each vertex x_i, let D_{x_i} be the component of $\langle M^- \cup M_1 \rangle$ containing x_i. Since C_1 is an SPU, D_{x_i} is a cycle and $V(D_{x_i}) \cap V(C_1) = \{x_i\}$. If t is even, we do x_1x_2-switch, …, $x_{t-1}x_t$-switch w.r.t. M_1 and let $M_{1,1} = M_1 + \{x_1x_2, x_3x_4, \ldots, x_{t-1}x_t\} - \bigcup_{i=1}^{t} \{x_i x_i^*\}$. By Claim 3.5, every component of $\langle M^- \cup M_{1,1} \rangle$ sharing a vertex with C_1 is a path except for D_1. If t is odd, we do v_1x_1-switch, $\ldots, x_{t-1}x_t$-switch w.r.t. M_1 and let $M_{1,1} = M_1 + \{v_1x_1, x_2x_3, \ldots, x_{t-1}x_t\} - \{v_1 v_1^*\} - \bigcup_{i=1}^{t} \{x_i x_i^*\}$. By Claim 3.5, every component of $\langle M^- \cup M_{1,1} \rangle$ sharing a vertex with C_1 is a path. For both parities of t, since $w_m \in V(C_1) = V(C_{m+1})$ and $w_m \neq v_1$, it follows that the cycle component D_m become a path component of $\langle M^- \cup M_{1,1} \rangle$.

Let $C_2 = v_2 y_1 \ldots y_s v_2$. Similarly to the discussion above, after a sequence of uv-switch in C_2, we get $M_{1,2}$ from $M_{1,1}$ such that every component of $\langle M^- \cup M_{1,2} \rangle$ sharing a vertex with C_2 is a path except for D_2. Since $w_1 \in V(C_2)$ and $w_1 \neq v_2$, it follows that the cycle component D_1 becomes a path component in $\langle M^- \cup M_{1,2} \rangle$ if it was unchanged when we worked on C_1.

Continuing this procedure in such way, we get $M_{1,m}$ from $M_{1,m-1}$ such that every component of $\langle M^- \cup M_{1,m} \rangle$ sharing a vertex with C_m is a path except for D_m. But, D_m has become a part of a path component in $\langle M^- \cup M_{1,1} \rangle$. In summary, all cycles M_1-good cycles $C_1, \ldots, C_m$ become $M_{1,m}$-good. Note that $E(C^*) \cap M_1^* = E(C^*) \cap M_1$ for any $C^* \in \mathcal{C} \setminus \{C_1, \ldots, C_m\}$ and $M_1 \cap E_{F_1}(v) = \{vv^*\}$ for any $v \in V(C_1 \cup \ldots \cup C_m)$. It follows that $M_{1,m}$ is a feasible matching and all M_1-good cycles are $M_{1,m}$-good, i.e., $M_1 \prec M_1^*$, giving a contradiction to the maximality of M_1, which completes our proof. $\square$

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